

# LIFTING MONOMIAL IDEALS

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**ABSTRACT.** We show how to lift any monomial ideal  $J$  in  $n$  variables to a saturated ideal  $I$  of the same codimension in  $n + t$  variables. We show that  $I$  has the same graded Betti numbers as  $J$  and we show how to obtain the matrices for the resolution of  $I$ . The cohomology of  $I$  is described. Making general choices for our lifting, we show that  $I$  is the ideal of a reduced union of linear varieties with singularities that are “as small as possible” given the cohomological constraints. The case where  $J$  is Artinian is the nicest. In the case of curves we obtain stick figures for  $I$ , and in the case of points we obtain certain  $k$ -configurations which we can describe in a very precise way.

## CONTENTS

1. Introduction	1
2. Lifting Monomial Ideals: Most General Case	3
3. Configurations of Linear Varieties	13
4. Pseudo-Liftings and Liftings of Artinian monomial ideals	18
References	25

## 1. INTRODUCTION

Let  $J \subset K[X_1, \dots, X_n]$  be the ideal of a subscheme,  $W$ , of projective space  $\mathbb{P}^{n-1}$ . An important question in general is to determine what subschemes  $V$  of  $\mathbb{P}^n$  have  $W$  as hyperplane section. More generally, we seek subschemes of  $\mathbb{P}^{n+k}$  with  $W$  as the intersection with a general linear space of complementary dimension. Furthermore, one would like to find  $V$  with nice properties, and to describe the minimal free resolution of  $V$ . If  $W$  is an arithmetically Cohen-Macaulay subscheme of  $\mathbb{P}^{n-1}$  of dimension  $\geq 1$ , our construction should give the same property for  $V$ .

It will be convenient to use new letters for the additional variables. Hence algebraically, in the hyperplane section case we seek a saturated ideal  $I \subset K[X_1, \dots, X_n, u_1]$  so that  $J = (I, u_1)/(u_1)$ . In this context, the problem makes sense even if  $J$  is not saturated (or even if  $J$  is Artinian so there is no  $W$ ). In the more general setting, we seek  $I \subset K[X_1, \dots, X_n, u_1, \dots, u_t]$  so that  $J = (I, u_1, \dots, u_t)/(u_1, \dots, u_t)$ . This is a special case of the so-called “lifting problem.”

An obvious solution is to simply view  $J \subset K[X_1, \dots, X_{n+1}]$  and consider the cone over  $W$ . This has exactly the same resolution as  $J$ , but a nasty singularity at the vertex point.

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*Date:* July 6, 1999.

\* Partially supported by the Department of Mathematics of the University of Paderborn

\*\* Partially supported by the Department of Mathematics of the University of Notre Dame .

One solution, in the case where  $W$  is a curvilinear zeroscheme, can be found in [1] and [25], where they show that  $V$  can be taken to be even a smooth curve. However, the curves obtained are almost never arithmetically Cohen-Macaulay, and in [26] it is shown that “most of the time” there does not even exist a smooth arithmetically Cohen-Macaulay curve with  $W$  as hyperplane section.

A better solution, from our point of view, is for the case of codimension two varieties. In this case the arithmetically Cohen-Macaulay property means that the minimal free resolution of  $J$  is a short exact sequence, so in this case the only map to worry about is given by the Hilbert-Burch matrix (since the next map is given by the maximal minors of the Hilbert-Burch matrix). One can then add a new variable, say  $t$ , to each entry of the matrix in a sufficiently general way and obtain an arithmetically Cohen-Macaulay  $V$  with  $W$  as hyperplane section. This was done in [7] in the case where  $W$  is a finite set of points in  $\mathbb{P}^2$ , and they showed that if  $W$  is of a certain (very general) form then  $V$  will be a smooth curve.

When the resolution is longer, however, there are more matrices in the resolution and it is very difficult to add variables to each entry and still preserve exactness (or even the property of being a complex). Even finding an ideal of the right codimension that restricts to  $J$  (say by replacing each occurrence of a variable in a generator of  $J$  by a linear form involving the new variables  $u_1, \dots, u_t$ ) is difficult. This was done in [19] for the case of codimension three arithmetically Gorenstein schemes, where the original schemes were lifted to reduced irreducible ones, but as in the codimension two case the whole resolution depends only on one matrix, so the problem of preserving exactness came for free.

Failing smoothness or irreducibility, a desirable property for curves in  $\mathbb{P}^3$  is to be a so-called stick figure. The study of such curves, especially from the point of view of the Hilbert scheme, can be found in [18], where Hartshorne solved the long-standing Zeuthen problem. An extension of this property to codimension two varieties can be found in [3]. In the case of dimension one, such curves were constructed in [21] which were arithmetically Cohen-Macaulay, using a method different from that in this paper.

The goal of this paper is to give a complete solution to the lifting problem in the case where  $J$  is a monomial ideal and we add any number of variables. We lift to a reduced union of linear varieties (see below). We build on work of Hartshorne [17] and of Geramita, Gregory and Roberts [9]. In those papers the authors began with an Artinian monomial ideal  $J$  and produced a “lifted” ideal  $I$  which they showed was the saturated ideal of a reduced set of points. Of course once the new ideal is constructed, it is no longer monomial, so the process cannot be repeated. Our approach is to mimic the construction mentioned above, but to introduce any number of variables at one time. We make the procedure somewhat easier to follow by introducing a matrix, called the “lifting matrix,” which contains all the information of the lifting.

In section 2 we describe our lifting method and we make a detailed study of the minimal free resolution of the ideal we obtain, as well as a study of the cohomology. The main tools are Taylor’s free resolution of a monomial ideal [24] and the theorem of Buchsbaum and Eisenbud [6] on what makes a complex exact. A lemma of Buchsbaum and Eisenbud on lifting [5] is also useful. The main result of the section is Proposition 2.6, where we show that the graded Betti numbers of the lifted ideal are exactly the same as those of

$J$ , and we describe how the maps in the resolution are lifted from those of the minimal free resolution of  $J$ . As a consequence we get that  $I$  is a saturated ideal, and we obtain a great deal of cohomological information. For instance, the depth of the lifted ideal is large, and the cohomology of  $I$  is a “lifting” (in a precise sense) of that of  $J$ . This section requires almost nothing about the lifting matrix, and as a result gives almost nothing (except dimension) in the way of “nice” geometric properties of the scheme defined by the lifted matrix. We even observe in Remark 2.21 that the lifting matrix can be chosen in such a way that the new ideal is not, strictly speaking, a lifting but does still have the cohomological properties just described. This fact is utilized in section 4. We call such an ideal a “pseudo-lifting.”

In section 3 we assume that the entries of the lifting matrix are sufficiently general. We define an extension of the notion of stick figures to any dimension or codimension. The strongest condition is what we call a “generalized stick figure.” We prove in Theorem 3.4 that our schemes  $V$  obtained by using this general lifting matrix are not only reduced, but in fact the union of the components of any given dimension are generalized stick figures away from  $W$ . As a corollary, if  $J$  is Artinian then  $V$  is an arithmetically Cohen-Macaulay generalized stick figure. Furthermore, we can construct in this way an arithmetically Cohen-Macaulay generalized stick figure corresponding to any allowable Hilbert function (Corollary 3.7). Moreover, any monomial ideal is the initial ideal of a radical ideal, and any Artinian monomial ideal is the initial ideal of a radical ideal defining an arithmetically Cohen-Macaulay generalized stick figure.

In section 4 we give a more detailed description of the configurations of linear varieties obtained by lifting Artinian monomial ideals, both in the strict sense and in the more general sense of pseudo-liftings mentioned above. It turns out that the case of lex-segment Artinian monomial ideals gives a particularly nice special case. They produce certain so-called “ $k$ -configurations,” while the other Artinian monomial ideals produce only so-called “weak  $k$ -configurations.” Using our approach we suggest an extension of  $k$ -configurations to higher dimension.

It is a pleasure to acknowledge the many contributions of Robin Hartshorne in the area of this paper and in the broader field of Algebraic Geometry. We dedicate this paper to him on the occasion of his sixtieth birthday.

## 2. LIFTING MONOMIAL IDEALS: MOST GENERAL CASE

Let  $k$  be an infinite field and let  $S = K[X_1, \dots, X_n]$  and  $R = K[X_1, \dots, X_n, u_1, \dots, u_t]$ . Our techniques also often work over a finite field: see Remark 3.2.

We shall be interested in the general idea of “lifting” a monomial ideal. More generally, we have

**Definition 2.1.** Let  $R$  be a ring and let  $u_1, \dots, u_t$  be elements of  $R$  such that  $\{u_1, \dots, u_t\}$  forms an  $R$ -regular sequence. Let  $S = R/(u_1, \dots, u_t)$ . Let  $B$  be an  $S$ -module and let  $A$  be an  $R$ -module. Then we say  $A$  is a  $t$ -lifting of  $B$  to  $R$  if  $\{u_1, \dots, u_t\}$  is an  $A$ -regular sequence and  $A/(u_1, \dots, u_t)A \cong B$ . When  $t = 1$  we will sometimes just refer to  $A$  as a *lifting* of  $B$ .

The following lemma of Buchsbaum and Eisenbud [5] will be very useful.

**Lemma 2.2.** *Let  $R$  be a ring,  $x \in R$ , and  $S = R/(x)$ . Let  $B$  be an  $S$ -module, and let*

$$\mathcal{F} : F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0$$

*be an exact sequence of  $S$ -modules with  $\text{coker } \phi_1 \cong B$ . Suppose that*

$$\Gamma : G_2 \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} G_0$$

*is a complex of  $R$ -modules such that*

- (1) *The element  $x$  is a non-zero divisor on each  $G_i$ ,*
- (2)  *$G_i \otimes_R S = F_i$ , and*
- (3)  *$\psi_i \otimes_R S = \phi_i$ .*

*Then  $A = \text{coker } \psi_1$  is a lifting of  $B$  to  $R$ .*

In Definition 2.1, notice that if  $A$  is an ideal in a polynomial ring then  $A$  is a lifting of  $B$  if and only if  $x$  is not a zero-divisor on  $R/A$  and  $(A, x)/(x) \cong B$ . To set notation we rewrite Definition 2.1 for the case of ideals.

**Definition 2.3.** Let  $R = K[X_1, \dots, X_n, u_1, \dots, u_t]$  and  $S = K[X_1, \dots, X_n]$ . Let  $I \subset R$  and  $J \subset S$  be homogeneous ideals. Then we say  $I$  is a  *$t$ -lifting* of  $J$  to  $R$  (or when  $R$  is understood, simply a  *$t$ -lifting* of  $J$ ) if  $(u_1, \dots, u_t)$  is a regular sequence on  $R/I$  and  $(I, u_1, \dots, u_t)/(u_1, \dots, u_t) \cong J$ . We say that  $I$  is a *reduced  $t$ -lifting* of  $J$  if it is a  $t$ -lifting and if furthermore  $I$  is a radical ideal in  $R$ .

In this section we show how to lift a monomial ideal, adding any number of variables. Later we will make “general” selections so that the lifted ideals will be the saturated ideals of projective subschemes which are not only reduced but in fact *generalized stick figures* (see Definition 3.3).

In [9] the authors show, following an idea of Hartshorne [17], how to get a reduced 1-lifting of a monomial ideal. The ideal produced in this way is no longer a monomial ideal, so their construction cannot be repeated to produce higher lifting. Furthermore, although they do not explicitly say so, their monomial ideals seem to be Artinian, since they describe their lifted ideal geometrically as the ideal of polynomials vanishing on a certain set  $\bar{M}$ , and they note that  $u_1$  (in our notation) is not a zero divisor since it does not vanish at any element of  $\bar{M}$ , suggesting that  $\bar{M}$  is a finite set. In this section we will show how to  $t$ -lift a monomial ideal by in fact lifting the whole resolution. This gives a great deal of information about the lifted ideal.

Our approach to the lifting follows that of [9], and we now recall some notation introduced there. Let  $\mathbb{N} = \{0, 1, 2, \dots\}$  and if  $\alpha = (a_1, \dots, a_n) \in \mathbb{N}^n$ , we let  $X^\alpha = X_1^{a_1} \cdots X_n^{a_n}$ . Letting  $P$  denote the set of monomials in  $S$  (including 1), then this gives a bijection between  $P$  and  $\mathbb{N}^n$ . The set  $\mathbb{N}^n$  may be partially ordered by  $(a_1, a_2, \dots, a_n) \leq (b_1, b_2, \dots, b_n)$  if and only if  $a_i \leq b_i$  for all  $i$ . This partially ordering translates, via the bijection above, to divisibility of monomials in  $P$ .

For each variable  $X_j$ ,  $1 \leq j \leq n$ , choose infinitely many linear forms  $L_{j,i} \in K[X_j, u_1, \dots, u_t]$  ( $i = 1, 2, \dots$ ). In subsequent sections we will impose other conditions so that each  $L_{j,i}$  is chosen sufficiently generically with respect to  $L_{j,1}, \dots, L_{j,i-1}$ , but for this section we do not even assume that no  $L_{j,i}$  is a scalar multiple of an  $L_{j,k}$ . We only assume

that the coefficient of  $X_j$  in  $L_{j,i}$  is not zero. For any given example, only finitely many  $L_{j,i}$  need be chosen. In this case the matrix  $A = [L_{j,i}]$  will be called the *lifting matrix*:

$$A = \begin{bmatrix} L_{1,1} & L_{1,2} & L_{1,3} & \cdots \\ L_{2,1} & L_{2,2} & L_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \\ L_{n,1} & L_{n,2} & L_{n,3} & \cdots \end{bmatrix}.$$

To each monomial  $m \in P$ , if  $m = \prod_{j=1}^n X_j^{a_j}$ , we associate the homogeneous polynomial

$$(2.1) \quad \bar{m} = \prod_{j=1}^n \left( \prod_{i=1}^{a_j} L_{j,i} \right) \in R.$$

If  $J = (m_1, \dots, m_r)$  is a monomial ideal in  $S$  then we denote by  $I$  the ideal  $(\bar{m}_1, \dots, \bar{m}_r) \subset R$ . The crucial aspect of this construction is that at each occurrence of some power  $a$  of  $X_j$  in any monomial, we always take the product  $L_{j,1} \cdots L_{j,a}$  of the *first  $a$  entries* of the  $j$ -th row (corresponding to  $X_j$ ) of the lifting matrix. This redundancy is what makes the construction work.

**Remark 2.4.** Geometrically the choice of the  $L_{j,i}$  above amounts to viewing  $\mathbb{P}^{n-1} \subset \mathbb{P}^{n+t-1}$  and choosing hyperplanes in  $\mathbb{P}^{n+t-1}$  containing the hyperplane in  $\mathbb{P}^{n-1}$  defined by  $X_j$ .

**Remark 2.5.** The construction above is a generalization, as mentioned, of the construction of [9], which gave the case  $t = 1$ . We would like to also point out that the unpublished thesis of Schwartau [22] contains a construction which is also similar. That is, his notion of *polarization* of a monomial ideal is obtained by replacing the repeated variables in a monomial ideal by *new variables* instead of by different linear forms in one or  $t$  new variables. His conclusions are different from ours, although some of his preparatory lemmas are useful and are quoted below.

We now show how to lift the minimal free resolution of a monomial ideal. The first step is to use Taylor's (not necessarily minimal) free resolution [24]. Recall that the entries of the matrices of such a resolution are themselves monomials (with a suitable choice of basis). These entries will be replaced by successive copies of the  $L_{j,i}$  in “almost” the analogous way to what was done for the generators.

**Proposition 2.6.** *Let  $J = (m_1, \dots, m_r) \subset S$  be a monomial ideal and let  $I = (\bar{m}_1, \dots, \bar{m}_r)$  be the ideal described above. Consider a (minimal) free  $S$ -resolution*

$$0 \rightarrow F_p \xrightarrow{\phi_p} F_{p-1} \xrightarrow{\phi_{p-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} J \rightarrow 0$$

*of  $J$ . Then  $I$  has a (minimal) free  $R$ -resolution*

$$(2.2) \quad 0 \rightarrow \bar{F}_p \xrightarrow{\bar{\phi}_p} \bar{F}_{p-1} \xrightarrow{\bar{\phi}_{p-1}} \cdots \xrightarrow{\bar{\phi}_2} \bar{F}_1 \xrightarrow{\bar{\phi}_1} I \rightarrow 0$$

*where  $\bar{F}_i$  is a “lifting” of  $F_i$  in the obvious way and the maps  $\bar{\phi}_i$  are “liftings,” explicitly obtained from the  $\phi_i$  as described below in the proof.*

*Proof.* For the convenience of the reader, we first briefly recall from [8] the relevant facts about Taylor's resolution. Let  $m_1, \dots, m_r$  be monomials in  $S$ . Taylor's free resolution of the ideal  $J = (m_1, \dots, m_r)$  has the form

$$0 \rightarrow F_r \xrightarrow{d_r} \dots \xrightarrow{d_2} F_1 \xrightarrow{d_1} J \rightarrow 0$$

where the free modules and maps are defined as follows. Let  $F_s$  be the free module on basis elements  $e_A$ , where  $A$  is a subset of length  $s$  of  $\{1, \dots, r\}$ . Set

$$m_A = \text{lcm}\{m_i | i \in A\}.$$

For each pair  $A, B$  such that  $A$  has  $s$  elements and  $B$  has  $s-1$  elements, let  $A = \{i_1, \dots, i_s\}$  and suppose that  $i_1 < \dots < i_s$ . Define

$$c_{A,B} = \begin{cases} 0 & \text{if } B \not\subset A \\ (-1)^k m_A / m_B & \text{if } A = B \cup \{i_k\} \text{ for some } k. \end{cases}$$

Then we define  $d_s : F_s \rightarrow F_{s-1}$  by sending  $e_A$  to  $\sum_B c_{A,B} e_B$ . Clearly Taylor's resolution does not, in general, have the same length as the minimal free resolution of  $J$  since the length is equal to the number of generators of  $J$  and, for example,  $F_r$  has rank 1.

Now, a key observation is given in [8] at the end of Exercise 17.11 (page 439): the exact same construction can be used in a much more general setting to produce at least a complex. In particular, in our situation, replacing the monomial ideal  $J = (m_1, \dots, m_r) \subset S$  by the ideal  $I = (\bar{m}_1, \dots, \bar{m}_r) \subset R$ , we see that we at least have a complex. Furthermore, since  $m_A$  is also a monomial it can be lifted as above, and one can immediately check that we have

$$(2.3) \quad \bar{m}_A = \text{lcm}\{\bar{m}_i | i \in A\}.$$

Note that we do not necessarily have the same kind of lifting for the  $c_{A,B}$ ; it is a lifting, but the products of  $L_{j,i}$  do not necessarily begin with  $L_{j,1}$ , as we indicated above (see Example 2.7). However, this does not matter. We conclude that we have a complex

$$0 \rightarrow \bar{F}_r \xrightarrow{\bar{d}_r} \dots \xrightarrow{\bar{d}_2} \bar{F}_1 \xrightarrow{\bar{d}_1} I \rightarrow 0$$

where the  $\bar{d}_k$  restrict to the  $d_k$ .

It remains to check that this complex is in fact a resolution for  $I$ . By the Buchsbaum-Eisenbud exactness criterion, we have to show that (i)  $\text{rank } \bar{d}_{k+1} + \text{rank } \bar{d}_k = \text{rank } \bar{F}_k$  for all  $k$  and (ii) the ideal of  $(\text{rank } \bar{d}_k)$ -minors of  $\bar{d}_k$  contains a regular sequence of length  $k$ , or is equal to  $R$ . Both of these follow immediately from the fact that the restriction of the  $\bar{d}_k$  is  $d_k$  for each  $k$ , and we know that the restriction is Taylor's resolution and hence satisfies these two properties.

Finally, an entry of one of the  $d_k$  is 1 if and only if the corresponding lifted matrix has a 1 in the same position. Hence also the minimal free resolutions agree, as claimed.  $\square$

**Example 2.7.** Let  $n = 3$  and  $t = 2$ . Consider the ideal  $J = (X_1^2 X_2, X_2^2 X_3, X_3^2 X_1)$ . This has the minimal free  $S$ -resolution

$$0 \rightarrow S(-6) \xrightarrow{\phi_3} S(-5)^3 \xrightarrow{\phi_2} S(-3)^3 \xrightarrow{\phi_1} J \rightarrow 0$$

where

$$\begin{aligned}\phi_1 &= \begin{bmatrix} X_1^2 X_2 & X_2^2 X_3 & X_3^2 X_1 \end{bmatrix} \\ \phi_2 &= \begin{bmatrix} -X_3^2 & 0 & -X_2 X_3 \\ 0 & -X_1 X_3 & X_1^2 \\ X_1 X_2 & X_2^2 & 0 \end{bmatrix} \\ \phi_3 &= \begin{bmatrix} -X_2 \\ X_1 \\ X_3 \end{bmatrix}\end{aligned}$$

Notice that in this case Taylor's resolution is minimal. In order to lift, we choose linear forms for each variable as follows:

$$\begin{aligned}X_1 &: L_{1,1}(X_1, u_1, u_2), L_{1,2}(X_1, u_1, u_2), \dots \\ X_2 &: L_{2,1}(X_2, u_1, u_2), L_{2,2}(X_2, u_1, u_2), \dots \\ X_3 &: L_{3,1}(X_3, u_1, u_2), L_{3,2}(X_3, u_1, u_2), \dots\end{aligned}$$

Then we set  $I = (L_{1,1}L_{1,2}L_{2,1}, L_{2,1}L_{2,2}L_{3,1}, L_{3,1}L_{3,2}L_{1,1})$  and its minimal free resolution has the form

$$0 \rightarrow R(-6) \xrightarrow{\bar{\phi}_3} R(-5)^3 \xrightarrow{\bar{\phi}_2} R(-3)^3 \xrightarrow{\bar{\phi}_1} I \rightarrow 0$$

where

$$\begin{aligned}\bar{\phi}_1 &= \begin{bmatrix} L_{1,1}L_{1,2}L_{2,1} & L_{2,1}L_{2,2}L_{3,1} & L_{3,1}L_{3,2}L_{1,1} \end{bmatrix} \\ \bar{\phi}_2 &= \begin{bmatrix} -L_{3,1}L_{3,2} & 0 & -L_{2,2}L_{3,1} \\ 0 & -L_{1,1}L_{3,2} & L_{1,1}L_{1,2} \\ L_{1,2}L_{2,1} & L_{2,1}L_{2,2} & 0 \end{bmatrix} \\ \bar{\phi}_3 &= \begin{bmatrix} -L_{2,2} \\ L_{1,2} \\ L_{3,2} \end{bmatrix}\end{aligned}$$

Notice, for example, that the entries of  $\bar{\phi}_3$  are all of the form  $L_{j,2}$  rather than  $L_{j,1}$ .

**Corollary 2.8.**  $\text{depth } R/I = \text{depth } S/J + t$ . In particular,  $\text{depth } R/I \geq t$ .

*Proof.*

$$\begin{aligned}\text{depth } R/I &= n + t - \text{pd}_R R/I \\ &= n + t - \text{pd}_S S/J \\ &= n + t - [n - \text{depth } S/J]\end{aligned}$$

□

For the rest of this paper we will be interested in the projective subschemes defined by  $I$  and  $J$ :

We denote by  $V$  the scheme in  $\mathbb{P}^{n+t-1}$  defined by  $I$ , and by  $W$  the scheme in  $\mathbb{P}^{n-1}$  defined by  $J$ . If this latter is empty, i.e. if  $S/J$  is Artinian, then for the purposes below we formally define  $\text{codim } W = n$  and  $\deg W = \text{length } S/J$ .

**Lemma 2.9.**  $V$  has the same codimension in  $\mathbb{P}^{n+t-1}$  that  $W$  has in  $\mathbb{P}^{n-1}$ .

*Proof.* Suppose that the codimension of  $W$  is  $c$ . Clearly  $W$  is the intersection in  $\mathbb{P}^{n+t-1}$  of  $V$  with the codimension  $t$  linear space defined by  $u_1 = \dots = u_t = 0$ . Hence  $\text{codim } V \geq c$ . So we only have to prove  $\text{codim } V \leq c$ .

Since  $J$  is a monomial ideal, all associated primes are of the form  $(X_{i_1}, \dots, X_{i_k})$ . By hypothesis, then, there exist  $X_{i_1}, \dots, X_{i_c}$  such that every element of  $J$  is in the ideal  $(X_{i_1}, \dots, X_{i_c})$ . By the construction of  $I$  it is clear that every element of  $I$  is in the ideal  $(L_{i_1,1}, \dots, L_{i_c,1})$ . Hence  $\text{codim } V \leq c$  as claimed.  $\square$

**Corollary 2.10.** (i) *The ideal  $I$  is saturated.*

(ii)  *$S/J$  is Cohen-Macaulay (including the case where it is Artinian) if and only if  $R/I$  is Cohen-Macaulay.*

(iii)  *$(I, u_1, \dots, u_t)/(u_1, \dots, u_t) \cong J$ .*

(iv)  *$\deg V = \deg W$ .*

(v)  *$(u_1, \dots, u_t)$  is a regular sequence on  $A = R/I$ .*

*Combining (iii) and (v), we get that  $I$  is a  $t$ -lifting of  $J$ .*

*Proof.* (i) is immediate from Corollary 2.8. For (ii),  $\Rightarrow$  follows from Corollary 2.8 and Lemma 2.9 (the converse is immediate). (iii) is obvious. (iv) follows from Proposition 2.6 and a computation of the Hilbert polynomial.

To prove (v) we use induction on  $t$ . The case  $t = 1$  follows from Proposition 2.6 and Lemma 2.2. Let  $I_i$  ( $1 \leq i \leq t$ ) denote the ideal in  $R_i := K[X_1, \dots, X_n, u_1, \dots, u_i]$  which lifts  $J$  to that ring via our construction. Note  $I = I_t$ ; in this case we continue to refer to the ideal as  $I$ . By induction we may assume that  $(u_1, \dots, u_{t-1})$  is a regular sequence on  $R_{t-1}/I_{t-1}$ . Then by Proposition 2.6 we see that setting  $R = K[X_1, \dots, X_n, u_1, \dots, u_t]$ ,  $x = u_t$ ,  $S = K[X_1, \dots, X_n, u_1, \dots, u_{t-1}]$  and  $B = S/I_{t-1}$ , all the hypotheses of Lemma 2.2 are satisfied, and hence  $R/I$  is a lifting of  $R_{t-1}/I_{t-1}$  to  $R$ . The result follows immediately.  $\square$

The following result from [4] (Lemma 3.1.16, p. 94) will be useful. Here we give the graded version.

**Lemma 2.11.** *Let  $R$  be a graded ring, and let  $M$  and  $N$  be graded  $R$ -modules. If  $x$  is a homogeneous  $R$ - and  $M$ -regular element with  $x \cdot N = 0$ , then*

$$\text{Ext}_R^{i+1}(N, M)(-\deg x) \cong \text{Ext}_{R/(x)}^i(N, M/xM)$$

*for all  $i \geq 0$ .*

**Proposition 2.12.** *If  $\text{Ext}_S^i(S/J, S) \neq 0$  for some  $i \in \mathbb{Z}$  then  $\text{Ext}_R^i(R/I, R)$  is a  $t$ -lifting of  $\text{Ext}_S^i(S/J, S)$ . In particular,  $\text{depth } \text{Ext}_R^i(R/I, R) = t + \text{depth}_S^i(S/J, S)$ .*

*Proof.* We adopt the following notation:

$$\begin{aligned} (-)^* &= \text{Hom}_R(-, R) \text{ (dualizing with respect to } R) \\ (-)^\vee &= \text{Hom}_S(-, S) \text{ (dualizing with respect to } S) \end{aligned}$$



Let  $t = 1$  and put  $u = u_1$ . Let

$$\mathbb{F}_\bullet : \quad \cdots \rightarrow F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_1} S \rightarrow S/J \rightarrow 0$$

be the Taylor resolution of  $S/J$ . From the proof of Proposition 2.6 we know that  $R/I$  has a resolution

$$\bar{\mathbb{F}}_\bullet : \quad \cdots \rightarrow \bar{F}_{i+1} \xrightarrow{\bar{\varphi}_{i+1}} \bar{F}_i \xrightarrow{\bar{\varphi}_i} \cdots \xrightarrow{\bar{\varphi}_1} R \rightarrow R/I \rightarrow 0$$

such that  $\bar{\mathbb{F}}_\bullet \otimes_R S = \mathbb{F}_\bullet$ . In other words, we have a short exact sequence of complexes

$$0 \rightarrow \bar{\mathbb{F}}_\bullet(-1) \xrightarrow{u} \bar{\mathbb{F}}_\bullet \rightarrow \mathbb{F}_\bullet \rightarrow 0.$$

Dualizing we obtain the short exact sequences of complexes

$$0 \rightarrow \bar{\mathbb{F}}_\bullet^*(-1) \xrightarrow{u} \bar{\mathbb{F}}_\bullet^* \rightarrow \text{Ext}_R^1(\mathbb{F}_\bullet, R)(-1) \rightarrow 0,$$

since

$$0 \rightarrow \bar{F}_i(-1) \xrightarrow{u} \bar{F}_i \rightarrow F_i \rightarrow 0$$

induces

$$\begin{array}{ccccccc} 0 \rightarrow \text{Hom}_R(F_i, R) & \rightarrow & \bar{F}_i^* & \rightarrow & \bar{F}_i^*(1) & \rightarrow & \text{Ext}_R^1(F_i, R) \rightarrow \text{Ext}_R^1(\bar{F}_i, R). \\ \parallel & & & & & & \parallel \\ 0 & & & & & & 0 \end{array}$$

Lemma 2.11 gives

$$\text{Ext}_R^1(\mathbb{F}_\bullet, R)(-1) \cong \text{Hom}_S(\mathbb{F}_\bullet, S) = \mathbb{F}_\bullet^\vee.$$

Thus we get the short exact sequence of complexes

$$(2.4) \quad 0 \rightarrow \bar{\mathbb{F}}_\bullet^*(-1) \xrightarrow{u} \bar{\mathbb{F}}_\bullet^* \rightarrow \mathbb{F}_\bullet^\vee \rightarrow 0.$$

From the exact sequence

$$0 \rightarrow R/I(-1) \xrightarrow{u} R/I \rightarrow S/J \rightarrow 0$$

and using Lemma 2.11 and twisting by  $-1$  we get the induced long exact homology sequence

$$\begin{aligned} \cdots \rightarrow \text{Ext}_S^{i-1}(S/J, S) &\rightarrow \text{Ext}_R^i(R/I, R)(-1) \xrightarrow{u} \text{Ext}_R^i(R/I, R) \rightarrow \\ &\text{Ext}_S^i(S/J, S) \rightarrow \text{Ext}_R^{i+1}(R/I, R)(-1) \xrightarrow{u} \cdots \end{aligned}$$

**Claim 2.13.** *The multiplication map  $\text{Ext}_R^i(R/I, R)(-1) \xrightarrow{u} \text{Ext}_R^i(R/I, R)$  is injective for all  $i \geq 1$ .*

The claim immediately proves the assertion of this proposition since it allows us to break the homology sequence into short exact sequences of the form we want.

In order to show the claim we consider the commutative diagram

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
0 & \rightarrow & \text{im}(\bar{\varphi}_i^*)(-1) & \rightarrow & \bar{F}_i^*(-1) & \rightarrow & \bar{F}_i^*/\text{im}(\bar{\varphi}_i^*)(-1) \rightarrow 0 \\
& & \downarrow u & & \downarrow u & & \downarrow u \\
0 & \rightarrow & \text{im}(\bar{\varphi}_i^*) & \rightarrow & \bar{F}_i^* & \rightarrow & \bar{F}_i^*/\text{im}(\bar{\varphi}_i^*) \rightarrow 0 \\
& & & & \downarrow & & \\
& & & & F_i^\vee & & \\
& & & & \downarrow & & \\
& & & & 0 & & 
\end{array}$$

where the center column comes from (2.4). Denote by  $\alpha$  the multiplication on the left-hand side. Then the snake lemma provides a homomorphism  $\tau : \text{coker } \alpha \rightarrow F_i^\vee$ . Since  $\bar{F}_\bullet \otimes_R S = F_\bullet$  we have  $\text{coker } \alpha \cong \text{im}(\varphi_i^\vee)$ . Thus  $\tau$  is induced from the embedding  $\text{im}(\varphi_i^\vee) \hookrightarrow F_i^\vee$ , i.e.  $\tau$  is injective. Therefore the snake lemma shows that

$$\bar{F}_i^*/\text{im}(\bar{\varphi}_i^*)(-1) \xrightarrow{u} \bar{F}_i^*/\text{im}(\bar{\varphi}_i^*)$$

is injective. Since  $\text{Ext}_R^i(R/I, R)$  is a submodule of  $\bar{F}_i^*/\text{im}(\bar{\varphi}_i^*)$ , our claim is proved.

Now let  $t > 1$ . In the argument above, we have not used the specific lifting from  $J$  to  $I$ , but only the fact that a resolution of  $J$  lifts to a resolution of  $I$ . Hence our assertion follows by induction on  $t$ .  $\square$

**Corollary 2.14.**  *$V$  is either arithmetically Cohen-Macaulay (if  $S/J$  is Cohen-Macaulay) or else it fails to be both locally Cohen-Macaulay and equidimensional.*

*Proof.* We saw in Corollary 2.10 that  $R/I$  is Cohen-Macaulay if and only if  $S/J$  is Cohen-Macaulay. Assume, then, that  $S/J$  is not Cohen-Macaulay. Suppose  $V$  has codimension  $c$ ; note that  $c \leq n - 1$ . It is enough to show that  $\text{Ext}_R^i(R/I, R)$  does not have finite length for some  $i$  in the range  $c + 1 \leq i \leq n$ . We have by assumption that  $\text{Ext}_S^i(S/J, S)$  is non-zero for some  $i$  in the range  $c + 1 \leq i \leq n$ . Hence the result follows immediately from Proposition 2.12.  $\square$

**Example 2.15.** Let  $n = 3$ ,  $t = 1$  and consider the ideal  $J$  in  $K[X_1, X_2, X_3]$  defined by  $(X_1^2, X_1X_2, X_1X_3, X_2^2, X_2X_3)$ . Notice that  $J$  is not saturated, but that its saturation  $\tilde{J}$  is just  $(X_1, X_2)$ , hence  $S/\tilde{J}$  is Cohen-Macaulay but  $J$  has an irrelevant primary component. One can check that the lifted ideal  $I \subset K[X_1, X_2, X_3, u_1]$  is the saturated ideal of the union of a line  $\lambda \subset \mathbb{P}^3$  (defined by  $X_1 = X_2 = 0$ ) and two points. This is locally Cohen-Macaulay but not equidimensional. Its top dimensional part is arithmetically Cohen-Macaulay.

Now let  $n = 4$  and  $t = 1$  and consider the ideal  $J$  in  $K[X_1, X_2, X_3, X_4]$  defined by  $(X_1X_3, X_1X_4, X_2X_3, X_2X_4)$ .  $J$  is the saturated ideal of the disjoint union of two lines in  $\mathbb{P}^3$ , hence  $S/J$  is not Cohen-Macaulay. One can check that the lifted ideal  $I \subset K[X_1, X_2, X_3, X_4, u_1]$  is the saturated ideal of the union of two planes in  $\mathbb{P}^4$  meeting at a single point, which is equidimensional but not locally Cohen-Macaulay.

These examples also illustrate Lemma 2.18 and Theorem 3.4 below.

We conclude this section with some preparatory facts about monomial ideals and liftings. We will use them in the following section, where we make some generality assumptions on our lifting matrix, but they hold in the generality of the current section.

**Lemma 2.16.** *Let*

$$\begin{aligned} I^{(1)} &= (m_1^{(1)}, \dots, m_{n_1}^{(1)}) \\ &\vdots \\ I^{(\ell)} &= (m_1^{(\ell)}, \dots, m_{n_\ell}^{(\ell)}) \end{aligned}$$

*be monomial ideals in  $S$ . Then  $I^{(1)} \cap \dots \cap I^{(\ell)}$  is the ideal generated by*

$$\{\text{lcm}(m_{i_1}^{(1)}, \dots, m_{i_\ell}^{(\ell)})\}$$

*where the indices lie in the ranges*

$$\begin{aligned} 1 &\leq i_1 \leq n_1 \\ &\vdots \\ 1 &\leq i_\ell \leq n_\ell \end{aligned}$$

*Proof.* This is [22] Lemma 85. □

**Lemma 2.17.** *Let  $J_1, J_2 \subset S$  be monomial ideals and fix a lifting matrix  $A$ . For any monomial ideal  $J$ , denote by  $\bar{J}$  the lifting of  $J$  by  $A$ . Then*

- (i)  $J_1 \subset J_2$  if and only if  $\bar{J}_1 \subset \bar{J}_2$ .
- (ii)  $\overline{J_1 \cap J_2} = \bar{J}_1 \cap \bar{J}_2$ .

*Proof.* Part (i) is clear. For part (ii), the inclusion  $\subseteq$  follows from part (i). Equality will come by showing that the Hilbert functions are the same. Consider the sequences

$$\begin{aligned} 0 &\rightarrow J_1 \cap J_2 \rightarrow J_1 \oplus J_2 \rightarrow J_1 + J_2 \rightarrow 0 \\ 0 &\rightarrow \bar{J}_1 \cap \bar{J}_2 \rightarrow \bar{J}_1 \oplus \bar{J}_2 \rightarrow \bar{J}_1 + \bar{J}_2 \rightarrow 0 \end{aligned}$$

Note that  $\overline{J_1 + J_2} = \bar{J}_1 + \bar{J}_2$ . Thanks to Corollary 2.8 we then have the following calculation. (We use the notation  $\Delta^t h_{R/I}(x)$  for the  $t$ -th difference of the Hilbert function of  $R/I$ . This is standard notation, but in any case see the discussion at the end of section 3.)

$$\begin{aligned} \Delta^t h_{R/\overline{J_1 \cap J_2}}(x) &= h_{S/J_1 \cap J_2}(x) \\ &= h_{S/J_1}(x) + h_{S/J_2}(x) - h_{S/J_1 + J_2}(x) \\ &= \Delta^t h_{R/\bar{J}_1}(x) + \Delta^t h_{R/\bar{J}_2}(x) - \Delta^t h_{R/\bar{J}_1 + \bar{J}_2}(x) \\ &= \Delta^t h_{R/\bar{J}_1}(x) + \Delta^t h_{R/\bar{J}_2}(x) - \Delta^t h_{R/\bar{J}_1 + \bar{J}_2}(x) \\ &= \Delta^t h_{R/\bar{J}_1 \cap \bar{J}_2}(x) \end{aligned}$$

Hence the Hilbert functions agree. □

**Corollary 2.18.** *Let  $J \subset S$  be a monomial ideal and let  $I$  be the lifting of  $J$  using some lifting matrix  $A$ . Suppose that the primary decomposition of  $J$  is*

$$J = Q_1 \cap \dots \cap Q_r.$$

Then

$$I = \bar{Q}_1 \cap \cdots \cap \bar{Q}_r$$

where  $\bar{Q}_i$  is the ideal generated by the liftings of the generators of  $Q_i$ .

**Remark 2.19.** It is known (cf. [8] Exercise 3.8 or [22], proof of Theorem 91) that if  $J \subset S$  is a monomial ideal then we can write  $J = Q_1 \cap \cdots \cap Q_r$  where each  $Q_i$  is a complete intersection of the form  $(X_{i_1}^{a_1}, \dots, X_{i_p}^{a_p})$  with  $a_j \geq 1$  for all  $j$ ,  $1 \leq j \leq p$ .

**Corollary 2.20.** *With the convention that the empty set is equidimensional of dimension  $-1$ , we have*

- (i)  *$V$  is equidimensional if and only if  $W$  is equidimensional.*
- (ii) *If  $W$  is equidimensional then  $V$  is either arithmetically Cohen-Macaulay or not even locally Cohen-Macaulay.*

*Proof.* Part (i) follows from Corollary 2.18 and Remark 2.19. Part (ii) also uses Corollary 2.14 and its proof.  $\square$

**Remark 2.21.** In defining our lifting matrix  $A$ , we said that in this section we assume almost nothing about the linear forms which are its entries. We required only that the linear forms  $L_{j,i}$  from the  $j$ -th row were elements of the ring  $K[X_j, u_1, \dots, u_t]$ . But there is another direction that we can go with the techniques of this section. Let  $J$  be a monomial ideal in  $S = K[X_1, \dots, X_n]$  and let  $A$  be an  $n \times r$  matrix of linear forms so that  $r$  is at least as big as the largest power of a variable occurring in a minimal generator of  $J$ . However, now we choose the entries of  $A$  generically in  $R = K[X_1, \dots, X_n, u_1, \dots, u_t]$ , where we even allow  $t = 0$ . The degree of genericity we require is that the polynomials  $F_j = \prod_{i=1}^N L_{j,i}$ ,  $1 \leq j \leq n$ , define a complete intersection,  $X$ . Note that  $F_j$  is the product of the entries of the  $j$ -th row, and that the height of the complete intersection is  $n$ , the number of variables in  $S$ .

The same construction as before produces from  $J$  and  $A$  an ideal  $I$  of  $R$ . Now  $I$  will no longer be a lifting in the sense of Definition 2.1. However, we claim that Proposition 2.6 still holds, and hence so do any of the results of this section that do not have to do with lifting. As indicated in the proof of Proposition 2.6, the first step is to observe that in any case Taylor's resolution leads to a complex when we replace  $J$  by  $I$ . We again just have to show that the Buchsbaum-Eisenbud exactness criterion gives that we have a resolution. More precisely, we have to show that

- (i)  $\text{rank } \bar{d}_{k+1} + \text{rank } \bar{d}_k = \text{rank } \bar{F}_k$  for all  $k$ , and
- (ii) the ideal of  $(\text{rank } \bar{d}_k)$ -minors of  $\bar{d}_k$  contains a regular sequence of length  $k$ , or is equal to  $R$ .

Now, however, we do not have that the restriction of  $\bar{d}_k$  is  $d_k$ . Nevertheless, the matrices  $\bar{d}_k$  still have the same form as the  $d_k$ . The main thing to check is that  $\text{rank } d_k = \text{rank } \bar{d}_k$ .

The rank of such a matrix  $d_k$  is the largest number of linearly independent columns, so clearly  $\text{rank } \bar{d}_k \geq \text{rank } d_k$ . For the reverse inequality, the danger is that a linear combination of columns of  $d_k$  could be zero while the corresponding linear combinations of

columns of  $\bar{d}_k$  be non-zero. This could happen if  $0 \neq c_{AB} = c_{A'B}$ , where  $B = \{i_1, \dots, i_{s-1}\}$ ,  $A = B \cup \{i_k\}$  and  $A' = B \cup \{i'_k\}$ , but the analogous entries for  $I$  are not equal. Suppose

$$c_{AB} = (-1)^k \frac{\text{lcm}\{m_{i_1}, \dots, m_{i_{s-1}}, m_{i_k}\}}{\text{lcm}\{m_{i_1}, \dots, m_{i_{s-1}}\}} \quad \text{and} \quad c_{A'B} = (-1)^k \frac{\text{lcm}\{m_{i_1}, \dots, m_{i_{s-1}}, m_{i'_k}\}}{\text{lcm}\{m_{i_1}, \dots, m_{i_{s-1}}\}}$$

If  $m_{i_k}$  contributes to the lcm then it contains a power of a variable  $X_j$  which is larger than the powers of  $X_j$  in the other monomials  $m_{i_1}, \dots, m_{i_{s-1}}$ . Since  $c_{AB} = c_{A'B}$ ,  $m'_{i_k}$  contains the same power of  $X_j$ . Hence they contribute the same number of entries from the  $j$ -th row of the “lifting” matrix  $A$ , so the entries of the relevant columns of  $\bar{d}_k$  are equal.

From this fact, condition (i) follows immediately. Condition (ii) follows because any  $(\text{rank } d_k)$ -minor of  $d_k$  corresponds to a  $(\text{rank } \bar{d}_k)$ -minor of  $\bar{d}_k$ , and if  $k$  of the former form a regular sequence then clearly  $k$  of the latter do as well.

Thanks to Remark 2.21, we now extend the notion of lifting to a more general one. See also Theorem 4.7 and Corollary 4.9.

**Definition 2.22.** Let  $J$  be a monomial ideal in

$$S = K[X_1, \dots, X_n] \subset R = k[X_1, \dots, X_n, u_1, \dots, u_t],$$

where  $t \geq 0$ . Let  $A$  be an  $n \times r$  matrix of linear forms in  $R$ , where  $r$  is at least as big as the largest power of a variable occurring in a minimal generator of  $J$ , and such that the entries of  $A$  satisfy the condition that the polynomials  $F_j = \prod_{i=1}^N L_{j,i}$ ,  $1 \leq j \leq n$ , define a complete intersection,  $X$ . Let  $I$  be the ideal obtained from  $J$  by the construction described in this section. Then we shall call  $I$  a *pseudo-lifting* of  $J$ .

### 3. CONFIGURATIONS OF LINEAR VARIETIES

The results of the preceding section give a number of nice properties of the ideal  $I$  obtained from the monomial ideal  $J$ , with no assumption on the lifting matrix (or on  $J$  other than being a monomial ideal). In particular, Corollary 2.10 says that  $I$  is a  $t$ -lifting of  $J$ . We need one more fact in order to show that  $I$  is a *reduced*  $t$ -lifting of  $J$ , namely that  $I$  is radical. To get this, we need to make some assumptions on the lifting matrix. We will prove something more. We will prove not only that  $I$  defines the union,  $V$ , of linear varieties, but in fact we would like to control the way that these linear varieties intersect, and show that we can arrange that they intersect in a very nice way. For curves the ideal result would be to produce so-called *stick figures*, i.e. unions of lines such that no more than two pass through any given point.

Consider however the following example, which shows that stick figures are too ambitious in general, without some assumption on  $J$ , and also shows the approach we will take.

**Example 3.1.** Let  $n = 3$  and consider the ideal

$$J = (X_1^3, X_1^2 X_2, X_1^2 X_3, X_1 X_2^2, X_1 X_2 X_3, X_2^3, X_2^2 X_3).$$

Note that  $J$  is not saturated (see below) and that the saturation of  $J$  is not radical, defining instead a zeroscheme of degree 3 in  $\mathbb{P}^2$  supported on a point  $P$ . Let  $t = 1$ . For the lifting, we will have to make some “generality” assumption on the lifting matrix if we want to get a radical ideal. For example, if we took  $L_{j,i} = X_j$  for all  $1 \leq j \leq 3$  and

$1 \leq i \leq 3$ , we see that  $V$  is just a cone over the scheme  $W$  defined by  $J$ , hence not reduced.

Hence we will now assume that the linear forms  $L_{j,i} \in K[X_j, u_1]$  ( $1 \leq j \leq n$ ) are chosen generally. Then lifting we will check that we obtain the saturated ideal of the union,  $V$ , of three lines in  $\mathbb{P}^3$  passing through  $P$ , together with three distinct points in  $\mathbb{P}^3$ . In particular, the top dimensional part is not a stick figure.

Note that  $J = (X_1, X_2)^2 \cap (X_1, X_2, X_3)^3$ . Considering only the ideal  $(X_1, X_2)^2$ , we lift to the saturated ideal of just the union of the three lines passing through  $P$ . If we consider instead the lifting of  $(X_1, X_2, X_3)^3$ , we obtain instead the saturated ideal of 10 points in  $\mathbb{P}^3$ . However, 7 of these points lie on the union of the three lines, none at the vertex, so that taking the union (i.e. intersecting the ideals) gives the union of the three lines and three points described above. (Since 7 is not divisible by 3, this fact is somewhat surprising: it says that the three lines are not indistinguishable.)

To check this, following Remark 2.19 and removing redundant terms, notice that

$$(X_1, X_2)^2 = (X_1, X_2^2) \cap (X_1^2, X_2)$$

and

$$(X_1, X_2, X_3)^3 = (X_1, X_2^2, X_3^2) \cap (X_1^3, X_2, X_3) \cap (X_1, X_2, X_3^3) \cap (X_1, X_2^3, X_3) \cap (X_1^2, X_2^2, X_3) \cap (X_1^2, X_2, X_3^2).$$

In this form it is easy to see what the components of  $V$  will be after we lift: again we remove redundant terms and we obtain that  $(X_1, X_2)^2$  lifts to

$$(L_{1,1}, L_{2,1}) \cap (L_{1,1}, L_{2,2}) \cap (L_{1,2}, L_{2,1})$$

while  $(X_1, X_2, X_3)^3$  lifts to

$$(L_{1,1}, L_{2,1}, L_{3,1}) \cap (L_{1,1}, L_{2,1}, L_{3,2}) \cap (L_{1,1}, L_{2,2}, L_{3,1}) \cap (L_{1,1}, L_{2,2}, L_{3,2}) \cap (L_{1,2}, L_{2,1}, L_{3,1}) \cap (L_{1,3}, L_{2,1}, L_{3,1}) \cap (L_{1,1}, L_{2,1}, L_{3,3}) \cap (L_{1,1}, L_{2,3}, L_{3,1}) \cap (L_{1,2}, L_{2,2}, L_{3,1}) \cap (L_{1,2}, L_{2,1}, L_{3,2}).$$

Then  $J$  lifts to

$$(L_{1,1}, L_{2,1}) \cap (L_{1,1}, L_{2,2}) \cap (L_{1,2}, L_{2,1}) \cap (L_{1,3}, L_{2,1}, L_{3,1}) \cap (L_{1,1}, L_{2,3}, L_{3,1}) \cap (L_{1,2}, L_{2,2}, L_{3,1})$$

as claimed.

Example 3.1 shows that without some assumptions on  $J$  we cannot hope to get stick figures, but that we can obtain reduced unions of linear varieties. For the remainder of this paper we make the following convention (but see also Remark 3.2):

*For the lifting matrix  $A = [L_{j,i}]$ , we assume that on each row (corresponding to a variable  $X_j$ ), each entry  $L_{j,i}$  is chosen generically with respect to the preceding entries  $L_{j,1}, \dots, L_{j,i-1}$  on that row.*

Note, however, that for this section we continue to assume that  $L_{j,i} \in K[X_j, u_1, \dots, u_t]$ . In the next section we will use more generally chosen linear forms, thanks to Remark 2.21.

**Remark 3.2.** We would like to make somewhat more precise what we mean by “generically” in the above assumption. Geometrically, the condition is as follows: For any choice of  $k$  pairwise distinct entries  $L_1, \dots, L_k$  from the matrix, we require

$$(3.1) \quad \text{codim}(L_1, \dots, L_k) = \min\{n + t, k\}.$$

For any given monomial ideal  $J$ , note that this puts only a finite number of conditions on the lifting matrix. In fact, let  $N_k$  ( $1 \leq k \leq n$ ) be the largest power of  $X_k$  occurring as a factor of a minimal generator of  $J$ . Let  $N = \max\{N_1, \dots, N_n\}$ . Then the lifting matrix  $A = [a_{k,\ell}]$  for  $J$  can be chosen to have size  $n \times N$ , and we can further assume that

$$a_{k,\ell} = 0 \quad \text{if } \ell > N_k.$$

Then the above matrix condition is only required for non-zero entries of  $A$ .

We can describe a suitable matrix  $A$  explicitly. Put  $p = N_1 + \dots + N_n$  and choose  $p$  distinct elements  $b_1, \dots, b_p \in K$ . Consider the Vandermonde matrix

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ b_1 & b_2 & \cdots & b_p \\ \vdots & \vdots & \cdots & \vdots \\ b_1^t & b_2^t & \cdots & b_p^t \end{bmatrix}$$

Now let

$$\begin{aligned} B_1 &= \text{submatrix formed by the first } N_1 \text{ columns of } B \\ B_2 &= \text{submatrix formed by the next } N_2 \text{ columns of } B \\ &\vdots \\ B_n &= \text{submatrix formed by the next } N_n \text{ columns of } B \end{aligned}$$

Now we produce a lifting matrix  $A$ , of size  $n \times N$ , as follows. The first  $N_k$  entries of the  $k$ -th row are given by

$$(x_k, u_1, \dots, u_t) \cdot B_k.$$

If  $N_k < N$ , we “complete” the  $k$ -th row by zeros. Then the properties of the Vandermonde matrix ensure that the matrix condition (3.1) holds true for  $A$  if we choose only non-zero entries of  $A$ . It follows that we can lift the monomial ideal  $J$  to a “nice” ideal  $I$  provided our ground field  $K$  has at least  $p$  elements.

We have already seen in Example 3.1, and it will be made more precise in this section and especially in section 4, that the components of the scheme defined by the lifted ideal are linear varieties defined by suitable subsets of the entries of the lifting matrix (at most one entry from each row). Let  $\wp_1, \dots, \wp_j$  be associated prime ideals of the lifted ideal  $I$  of  $J$  using a lifting matrix  $A$ . Then it follows from (3.1) that

$$\text{codim}(\wp_1 + \dots + \wp_j) = \min\{n + t, \# \text{ entries of the lifting matrix occurring as minimal generators of some } \wp_i\}$$

The same holds for the explicit lifting matrix given above. Hence all the results below about generalized stick figures hold when the lifting matrix is this explicit matrix.

In the case  $t = 1$  and  $J$  Artinian, this matrix was essentially given in [9].

Inspired by Example 3.1, we now would like to examine the lifting more carefully. We would like to show, first, that we have a reduced  $t$ -lifting (cf. Definition 2.3). But in fact, we would like to see when we have a stick figure, for curves, and to extend this notion to higher dimension. In [3] the authors defined a “good linear configuration” to be a locally Cohen-Macaulay codimension two union of linear subspaces in  $\mathbb{P}^n$  such that the intersection of any three components has dimension at most  $n - 4$ . We would like to modify that definition slightly (removing the locally Cohen-Macaulay assumption and allowing arbitrary codimension), as follows. See also Remark 3.6.

**Definition 3.3.** Let  $V$  be a union of linear subvarieties of  $\mathbb{P}^m$  of the same dimension  $d$ . Then  $V$  is a generalized stick figure if the intersection of any three components of  $V$  has dimension at most  $d - 2$  (where the empty set is taken to have dimension  $-1$ ). In particular, if  $d = 1$  then  $V$  is a stick figure.

**Theorem 3.4.** Let  $J \subset S$  be a monomial ideal, let  $I$  be the  $t$ -lifting of  $J$  described in the paragraph preceding Remark 2.4, and let  $V$  (resp.  $W$ ) be the subscheme of  $\mathbb{P}^{n+t-1}$  defined by  $I$  (resp. the subscheme of  $\mathbb{P}^{n-1}$  defined by  $J$ ). Then

- (a)  $I$  is a radical ideal in  $R$ , i.e.  $I$  is a reduced  $t$ -lifting of  $J$ ; in fact,  $V$  is a union of linear varieties.
- (b) The union of the components of  $V$  of any given dimension  $d$  form a generalized stick figure away from  $W$ , in the sense that

$$(the\ intersection\ of\ any\ 3\ components) \setminus W_{red}$$

has dimension at most  $d - 2$ .

- (c) Suppose that  $J$  is Artinian. Then  $V$  is an arithmetically Cohen-Macaulay generalized stick figure.
- (d) Suppose that  $J$  is unmixed and not Artinian, and let  $J = Q_1 \cap \cdots \cap Q_r$  be a minimal primary decomposition of  $J$ . Let  $W_i$  be the scheme defined by  $Q_i$ , for each  $i$ . If  $t = 1$  then  $V$  is a generalized stick figure if and only if  $\deg W_i \leq 2$  for all  $i$ . If  $t \geq 2$  then  $V$  is a generalized stick figure.

*Proof.* We first prove that  $V$  is a union of linear varieties, and hence that  $I$  is radical. We know from Remark 2.19 that we can write  $J$  in the form  $J = Q_1 \cap \cdots \cap Q_r$  where each  $Q_i$  is a complete intersection of the form  $(X_{i1}^{a_1}, \dots, X_{ip}^{a_p})$ . Using Remark 2.4, it is clear that each lifted ideal  $\bar{Q}_i$  defines a reduced complete intersection in  $\mathbb{P}^{n+t-1}$ . Finally, from Lemma 2.18 we get that  $I$  is reduced and a union of linear varieties.

We have seen in Example 3.1 that we cannot hope that  $V$  will always be a generalized stick figure. Indeed, by Remark 2.4, if  $W$  is not empty then all components of  $V$  will pass through  $W$ , so if there are too many of these components then it will fail to be a generalized stick figure.

For (b), consider the lifting matrix

$$A = \begin{bmatrix} L_{1,1} & L_{1,2} & L_{1,3} & \cdots \\ L_{2,1} & L_{2,2} & L_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \\ L_{n,1} & L_{n,2} & L_{n,3} & \cdots \end{bmatrix}.$$



Consider a primary decomposition of  $J$  and write it as  $J = J_1 \cap J_2 \cap \cdots \cap J_s$  where, for  $1 \leq c \leq p$ ,  $J_c$  is the intersection of the primary components of codimension  $c$ . By Lemma 2.18 and Corollary 2.20, the components of  $V$  of codimension  $c$  are defined by the lifting  $I_c$  of  $J_c$ . Let  $W_c$  be the scheme defined by  $J_c$  (possibly empty if  $J_c$  is Artinian) and let  $V_c$  be the scheme defined by  $I_c$ .

Any component of  $V_c$  is a linear variety defined by  $c$  entries of the lifting matrix, no two from any given row, and it vanishes (set theoretically) on a component of  $W_c$ . Because the linear forms in  $A$  were chosen generally, subject only to the condition that the entries of the  $j$ -th row are linear forms in the variables  $X_j, u_1, \dots, u_t$ , it follows that the intersection of any 3 components has codimension 2 in  $V_c$ , at least away from the linear space  $\mathbb{P}^{n-1}$  defined by the vanishing of the variables  $u_1, \dots, u_n$ . (See also Remark 3.6.) But the intersection of  $V$  with this linear space is exactly  $W$ . This proves (b).

For (c), the fact that  $V$  is arithmetically Cohen-Macaulay follows from Corollary 2.10, while the fact that it is a generalized stick figure follows from (b), since  $W$  is empty.

For (d), let  $V_i$  be the lifting of  $W_i$ . Note that  $\deg V_i = \deg W_i$ , by Corollary 2.10 (iv),  $V$  is unmixed by Corollary 2.20, and hence  $V$  has exactly  $\deg W_i$  components passing through  $W_i$ . Furthermore,  $W_i$  has codimension  $t$  in  $V$ . Then using (b), we see that the only way that the intersection of three components of  $V$  could fail to have codimension two in  $V$  is if  $t = 1$  and  $\deg W_i \geq 3$  for some  $i$ .  $\square$

**Remark 3.5.** If  $J$  is Artinian, the fact that  $I$  is radical follows from [9]. Indeed, they show that a 1-lifting is a reduced set of points, and it is clear from the above that in the case of a  $t$ -lifting, a sequence of hyperplane sections reduces to a 1-lifting. If this 1-lifting is reduced then the original  $t$ -lifting must be reduced.

**Remark 3.6.** One might wonder why, in Definition 3.3, we do not define a generalized stick figure to involve the intersection of more than three components. As a first answer, let  $R$  be the polynomial ring in 5 variables and let  $A_1, A_2, A_3, A_4$  be general linear forms. Let  $F = A_1 \cdot A_2$  and let  $G = A_3 \cdot A_4$ . The complete intersection of  $F$  and  $G$  defines a union of four linear varieties of dimension 2. Any reasonable definition of a generalized stick figure must, in our opinion, include this example. However, the intersection of these four components has dimension 0 rather than being empty.

More generally, in the context of lifting, consider a lifting matrix

$$A = \begin{bmatrix} L_{1,1} & L_{1,2} & \cdots & L_{1,r} \\ L_{2,1} & L_{2,2} & \cdots & L_{2,r} \\ \vdots & \vdots & & \vdots \\ L_{n,1} & L_{n,2} & \cdots & L_{n,r} \end{bmatrix}$$

and let  $J = (X_1^{a_1}, X_2^{a_2}, X_3^{a_2}, \dots, X_n^{a_n})$  with  $a_1 \geq 2$  and  $a_2 \geq 2$ .  $J$  is Artinian, and the lifted ideal  $I$  defines a codimension  $n$  union of varieties,  $V$ , which contains in particular the components  $V_i$ ,  $1 \leq i \leq 4$ , defined by the ideals

$$\begin{aligned} I_{V_1} &= (L_{1,1}, L_{2,1}, L_{3,1}, \dots, L_{n,1}) \\ I_{V_2} &= (L_{1,1}, L_{2,2}, L_{3,1}, \dots, L_{n,1}) \\ I_{V_3} &= (L_{1,2}, L_{2,1}, L_{3,1}, \dots, L_{n,1}) \\ I_{V_4} &= (L_{1,2}, L_{2,2}, L_{3,1}, \dots, L_{n,1}). \end{aligned}$$

Assume that  $t \geq 3$ , so that  $\dim V = t - 1 \geq 2$  in  $\mathbb{P}^{n+t-1}$ . But then  $\dim V_1 \cap V_2 \cap V_3 \cap V_4 = t - 3$ , not  $t - 4$ . Hence for part (c) of Theorem 3.4 to be true, we must avoid intersecting four components.

For the last result of this section, we recall some basic facts about so-called “O-sequences,” which can be found in [13]. Macaulay showed that a sequence of non-negative integers  $\{c_i\}$ ,  $i \geq 0$ , can be the Hilbert function of a graded algebra  $A = S/I$  if and only if  $c_0 = 1$  and the  $c_i$  satisfy a certain growth condition (cf. [23]). Such a sequence is called an *O-sequence*. The sequence is said to be *differentiable* if the first difference sequence  $\{b_i\} = \Delta\{c_i\}$ , defined by  $b_i = c_i - c_{i-1}$ , is also an O-sequence. (We adopt the convention that  $c_{-1} = 0$ .) Extending this gives the notion of a *t-times differentiable O-sequence*, in case  $t$  successive differences  $\Delta^j\{c_i\}$ ,  $1 \leq j \leq t$ , are all still O-sequences. In [13] the authors showed that any differentiable O-sequence occurs as the Hilbert function of some graded algebra  $S/I$ , where  $I$  is *radical*.

An O-sequence  $\{c_i\}$  is said to have *dimension*  $d \geq 1$  (here “dimension” should be thought of as Krull dimension) if there is a non-zero polynomial  $f(x)$ , with rational coefficients, of degree  $d - 1$ , such that for all  $s \gg 0$ ,  $f(s) = c_s$ . If  $c_i = 0$  for all  $i \gg 0$  then we say that  $\{c_i\}$  has *dimension 0*. The sequence  $\{c_i\}$  is a *d-times differentiable O-sequence of dimension d* if and only if it is the Hilbert function of a Cohen-Macaulay algebra  $A = S/I$  of Krull dimension  $d$ . In this case the  $d$ -th successive difference  $\Delta^d\{c_i\}$  is the Hilbert function of the *Artinian reduction* of  $A$ . It is a finite sequence of positive integers, called the *h-vector* of  $A$  (cf. for instance [20]). By [13],  $I$  can be taken to be radical. We extend this with the following result, which was known in codimension two.

**Corollary 3.7.** *Let  $\{c_i\}$  be a t-times differentiable O-sequence of dimension t. Then  $\{c_i\}$  is the Hilbert function of an arithmetically Cohen-Macaulay generalized stick figure of dimension  $t - 1$  (as a subscheme of projective space). In particular, any 2-differentiable O-sequence of dimension 2 is the Hilbert function of some stick figure curve in projective space.*

Inverse Gröbner basis theory asks which monomial ideals arise as initial ideals of classes of ideals with prescribed properties. For example, it is conjectured that certain monomial ideals cannot be the initial ideal of a prime ideal. Instead, if we allow radical ideals, we have the following result.

**Corollary 3.8.** *Let  $J \subset S$  be a monomial ideal. Let  $>$  be the degree-lex order such that  $X_1 > \cdots > X_n > u_1 > \cdots > u_t$ . Then there is a radical ideal  $I$  in  $R$  such that the initial ideal of  $I$  with respect to  $>$  is  $J \cdot R$ . Moreover, if  $J$  is an Artinian monomial ideal in  $S$  then  $I$  can be chosen to be the defining ideal of an arithmetically Cohen-Macaulay generalized stick figure.*

*Proof.* Let  $I \subset R$  be a reduced  $t$ -lifting of  $J$ . Then we clearly have  $J \cdot R \subset \text{in}(I)$ . On the other hand we have for the Hilbert functions:  $\Delta^t h_{R/I}(j) = h_{S/J}(j) = \Delta^t h_{R/J \cdot R}(j)$ . Thus, the equality  $J \cdot R = \text{in}(I)$  follows.  $\square$

#### 4. PSEUDO-LIFTINGS AND LIFTINGS OF ARTINIAN MONOMIAL IDEALS

One of the main goals of this section is to describe the configurations of linear varieties which arise by lifting Artinian monomial ideals, in the sense of Definition 2.3. We give

the answer in Corollary 4.9. This is actually a special case of the more general notion of pseudo-lifting, as indicated in Remark 2.21, and we describe this situation in Theorem 4.7.

In this section we will consider a matrix

$$A = \begin{bmatrix} L_{1,1} & L_{1,2} & L_{1,3} & \cdots & L_{1,N} \\ L_{2,1} & L_{2,2} & L_{2,3} & \cdots & L_{2,N} \\ \vdots & \vdots & \vdots & & \vdots \\ L_{n,1} & L_{n,2} & L_{n,3} & \cdots & L_{n,N} \end{bmatrix}$$

of linear forms in  $R = K[X_1, \dots, X_n, u_1, \dots, u_t]$  satisfying the following genericity property, slightly more than we assumed in Remark 2.21:

Let  $F_j$  be the product of the entries of the  $j$ -th row of  $A$ . We assume that the ideal  $(F_1, \dots, F_n)$  defines a reduced complete intersection.

Throughout this section, “configuration” shall mean a reduced finite union of linear varieties all of the same dimension. We would like to describe geometrically the configurations which are the pseudo-liftings of Artinian monomial ideals, in the sense of Remark 2.21 and Definition 2.22. Recall that if the entries  $L_{j,i}$  are not in  $K[X_j, u_1, \dots, u_n]$ , these are not true liftings in the sense of Definition 2.3.

We have already seen in Theorem 3.4 (c) that these configurations will be arithmetically Cohen-Macaulay generalized stick figures. In the case of zero-dimensional schemes, the similarity of our construction to the notion of a  $k$ -configuration will be evident (but they are not quite the same), and we discuss the relation in Remark 4.10. Part of the discussion will involve the special case of Artinian lex-segment ideals, which we now recall.

**Definition 4.1.** Let  $>$  denote the degree-lexicographic order on monomial ideals, i.e.  $x_1^{a_1} \cdots x_n^{a_n} > x_1^{b_1} \cdots x_n^{b_n}$  if the first nonzero coordinate of the vector

$$\left( \sum_{i=1}^n (a_i - b_i), a_1 - b_1, \dots, a_n - b_n \right)$$

is positive. Let  $J$  be a monomial ideal. Let  $m_1, m_2$  be monomials in  $S$  of the same degree such that  $m_1 > m_2$ . Then  $J$  is a lex-segment ideal if  $m_2 \in J$  implies  $m_1 \in J$ .

Let  $J$  be an Artinian monomial ideal. Let  $N_j$  be the maximum power of  $X_j$  that occurs in a minimal generator of  $J$ , and let  $N$  be the maximum of the  $N_j$ . Consider the matrix

$$A = \begin{bmatrix} L_{1,1} & L_{1,2} & L_{1,3} & \cdots & L_{1,N} \\ L_{2,1} & L_{2,2} & L_{2,3} & \cdots & L_{2,N} \\ \vdots & \vdots & \vdots & & \vdots \\ L_{n,1} & L_{n,2} & L_{n,3} & \cdots & L_{n,N} \end{bmatrix}$$

where  $L_{j,i}$  is a generally chosen linear form in the ring  $R = K[X_1, \dots, X_n, u_1, \dots, u_t]$ . Let  $I$  be the ideal obtained from  $J$  using  $A$ , as in Remark 2.21, and let  $V$  be the configuration of linear varieties defined by the saturated ideal  $I$ . Note that the dimension of  $V$  as a projective subscheme is  $t - 1$ . Let  $F_1, \dots, F_n$  be defined by  $F_j = \prod_{i=1}^N L_{j,i}$ . Clearly  $F_j \in I$  for all  $j$ .

On the geometric side, the complete intersection  $(F_1, \dots, F_n)$  defines a union,  $X$ , of linear varieties of dimension  $t - 1$  and  $V$  is a subset of  $X$ . Each component of  $X$  is given by an  $n$ -tuple of linear forms  $(L_{1,i_1}, \dots, L_{n,i_n})$ . Let  $\Lambda \subset X$  be the component corresponding to  $(L_{1,i_1}, \dots, L_{n,i_n})$ . By abuse of notation we will write  $\Lambda = (L_{1,i_1}, \dots, L_{n,i_n}) \in V$  to mean that the corresponding linear variety is a component of  $V$ . Then recalling from (2.1) how the pseudo-lifting is defined, we see that

$$(4.1) \quad (L_{1,i_1}, \dots, L_{n,i_n}) \in V \Leftrightarrow \text{every monomial generator } m \in J \text{ is divisible by} \\ \text{at least one of } X_1^{i_1}, X_2^{i_2}, \dots, X_n^{i_n}.$$

As a result of (4.1) we immediately get the following description of the components of  $V$ , which is essentially contained in [9] in the case of dimension zero.

**Lemma 4.2.**  $(L_{1,i_1}, \dots, L_{n,i_n}) \in V$  if and only if  $X_1^{i_1-1} \dots X_n^{i_n-1} \notin J$ .

We would like to give a geometric description of the configurations that arise in this way.

Since  $J$  is an Artinian monomial ideal, it contains powers of all the variables  $X_j$ . Let  $\alpha_j$  be the least degree of  $X_j$  that appears in  $J$ . Without loss of generality, order and relabel the  $X_j$  so that  $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ .

Notice that, by (4.1), for  $i \geq 2$  we have that if the component  $(L_{1,i}, L_{2,i_2}, \dots, L_{n,i_n})$  is in  $V$  then the component  $(L_{1,i-1}, L_{2,i_2}, \dots, L_{n,i_n})$  is in  $V$ . More generally, we immediately see the following:

**Lemma 4.3.** For  $1 \leq j \leq n$  and  $i_j \geq 2$ , if  $(L_{1,i_1}, L_{2,i_2}, \dots, L_{j,i_j}, \dots, L_{n,i_n}) \in V$  then  $(L_{1,i_1}, L_{2,i_2}, \dots, L_{j,i_j-1}, \dots, L_{n,i_n}) \in V$ .

Furthermore, in the special case where  $J$  is a lex-segment ideal we have a stronger property:

**Lemma 4.4.** Let  $J$  be an Artinian monomial lex-segment ideal. For  $1 \leq j \leq n$  and  $i_j \geq 2$ , if  $(L_{1,i_1}, L_{2,i_2}, \dots, L_{j,i_j}, \dots, L_{n,i_n}) \in V$  then

$$(L_{1,i_1}, L_{2,i_2}, \dots, L_{j,i_j-1}, L_{j+1,i'_{j+1}}, \dots, L_{n,i'_n}) \in V$$

for any  $i'_{j+1}, \dots, i'_n \in \mathbb{N}$  such that  $i'_{j+1} + \dots + i'_n = i_{j+1} + \dots + i_n + 1$ .

*Proof.* Since  $(L_{1,i_1}, L_{2,i_2}, \dots, L_{j,i_j}, \dots, L_{n,i_n}) \in V$ , we have by Lemma 4.2 that

$$X_1^{i_1-1} \dots X_j^{i_j-1} \dots X_n^{i_n-1} \notin J.$$

But  $X_1^{i_1-1} \dots X_j^{i_j-1} \dots X_n^{i_n-1} > X_1^{i_1-1} \dots X_j^{i_j-2} X_{j+1}^{i'_{j+1}-1} \dots X_n^{i'_n-1}$ , so since  $J$  is a lex-segment ideal we get that

$$X_1^{i_1-1} \dots X_j^{i_j-2} X_{j+1}^{i'_{j+1}-1} \dots X_n^{i'_n-1} \notin J$$

as well. Hence

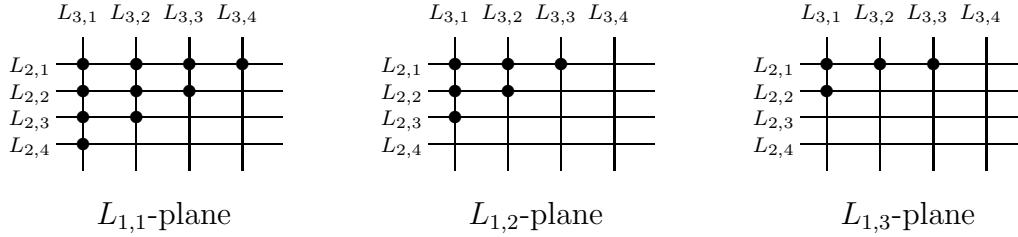
$$(L_{1,i_1}, L_{2,i_2}, \dots, L_{j,i_j-1}, L_{j+1,i'_{j+1}}, \dots, L_{n,i'_n}) \in V$$

as claimed.  $\square$

**Example 4.5.** Let

$$J = (X_1^3, X_1^2 X_2^2, X_1^2 X_2 X_3, X_1 X_2^3, X_1 X_2^2 X_3, X_1 X_2 X_3^2, X_1 X_3^3, X_2^4, X_2^3 X_3, X_2^2 X_3^2, X_2 X_3^3 X_3^4).$$

Note that  $J$  is “almost” a lex-segment ideal, missing only the monomial  $X_1^2 X_3^2$ . The  $h$ -vector of  $J$  is  $(1, 3, 6, 9, 1)$ . We first sketch the configuration obtained by pseudo-lifting  $J$ , by showing the configurations on the  $L_{1,1}$ -plane, the  $L_{1,2}$ -plane and the  $L_{1,3}$ -plane.

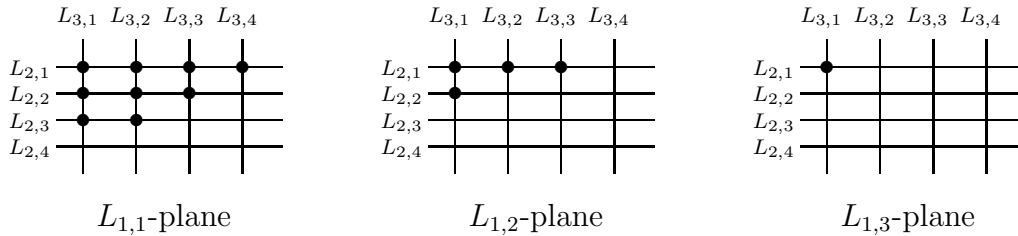


This satisfies the condition of Lemma 4.3, of course, but not Lemma 4.4. The “offending” point is  $(L_{1,3}, L_{2,1}, L_{3,3})$ , since the point  $(L_{1,2}, L_{2,1}, L_{3,4})$  is not in the configuration (take  $j = 1$  and  $k = 3$ ). Hence this configuration does not occur for lex-segment ideals.

**Example 4.6.** Here is a slightly more subtle example. Let  $n = 3$  and consider the monomial ideal

$$J = (X_1^3, X_1^2 X_2, X_1^2 X_3, X_1 X_2^2, X_1 X_2 X_3, X_2^3, X_1 X_3^3, X_2^2 X_3^2, X_2 X_3^3, X_3^4).$$

The component in degree 3 fails to be lex-segment because it is missing the monomial  $X_1 X_3^2$  after  $X_1 X_2 X_3$ . The configuration corresponding to the pseudo-lifting of  $J$  looks as follows:



At first glance one would be tempted to say that this satisfies the condition of Lemma 4.4, just from an examination of the picture. However, the fact that the point  $(L_{1,2}, L_{2,1}, L_{3,3})$  is in  $V$  requires that  $(L_{1,1}, L_{2,4}, L_{3,1}) \in V$  as well, and this does not hold.

We can now describe the configurations which arise as pseudo-liftings of Artinian monomial ideals and those which arise as pseudo-liftings of Artinian lex-segment monomial ideals.

**Theorem 4.7.** *Let  $V$  be a configuration of linear varieties of codimension  $n$  in  $\mathbb{P}^{n+t-1}$ . Let  $A$  be a matrix of linear forms,*

$$A = \begin{bmatrix} L_{1,1} & L_{1,2} & L_{1,3} & \cdots & L_{1,N} \\ L_{2,1} & L_{2,2} & L_{2,3} & \cdots & L_{2,N} \\ \vdots & \vdots & \vdots & & \vdots \\ L_{n,1} & L_{n,2} & L_{n,3} & \cdots & L_{n,N} \end{bmatrix}$$

*such that the polynomials  $F_j = \prod_{i=1}^N L_{j,i}$ ,  $1 \leq j \leq n$ , define a reduced complete intersection,  $X$ , containing  $V$ . Then  $V$  is the pseudo-lifting, via  $A$ , of an Artinian monomial ideal in  $n$  variables if and only if the condition of Lemma 4.3 holds. Furthermore,  $V$  is the pseudo-lifting, via  $A$ , of an Artinian lex-segment monomial ideal if and only if in addition the condition of Lemma 4.4 holds.*

*Proof.* For both parts of the theorem we have only to prove the sufficiency of the condition. We continue to use the same notation  $L_{j,i}$  for the linear form and for the corresponding hyperplane. Without loss of generality we can assume that  $V$  neither empty nor all of  $X$ .

Let  $I$  be the ideal generated by the set of all polynomials which are products of the form

$$\prod_{j=1}^n \left( \prod_{i=1}^{a_j} L_{j,i} \right)$$

and which vanish on  $V$ . Remove generators from this set to obtain a minimal generating set. Any generator of this form corresponds, via (2.1), to a monomial. Let  $J$  be the ideal generated by the set of all such monomials.  $J$  is Artinian since it contains a power of each of the variables, and clearly  $I$  is the pseudo-lifting of  $J$ . Hence  $I$  is the saturated ideal (thanks to Corollary 2.10) of a scheme  $Z$  with  $V \subseteq Z \subseteq X$ .

We have only to show that  $Z = V$ , and for this we use the condition of Lemma 4.3. Let  $\Lambda$  be a component of  $X$  which is not in  $V$  (OK since  $V$  is not all of  $X$ ). We will produce a polynomial  $F$  in  $I$  which does not vanish on  $\Lambda$ . Being a component of  $X$ ,  $\Lambda$  is of the form  $\Lambda = (L_{1,p_1}, \dots, L_{n,p_n})$  as above. Thanks to the condition of Lemma 4.3, we see that for each  $j$ , replacing the coordinate  $L_{j,p_j}$  by  $L_{j,p}$  for any  $p$  with  $p_j \leq p \leq N$  gives a component of  $X$  which is also not in  $V$ .

Let  $B_1$  be the set of indices  $j$  for which  $p_j = 1$  and let  $B_{\geq 2}$  be the set of indices  $j$  for which  $p_j \geq 2$ . Since  $V$  is not empty,  $B_{\geq 2}$  is not the empty set. Let

$$F = \prod_{j \in B_{\geq 2}} \left( \prod_{i=1}^{p_j-1} L_{j,i} \right)$$

Clearly  $F$  does not vanish on  $\Lambda$ . However, any component  $Q = (L_{1,q_1}, \dots, L_{n,q_n})$  of  $X$  which is which is not in the vanishing locus of  $F$  has entries which satisfy  $q_j \geq p_j$  for  $j \in B_{\geq 2}$  and  $q_j \geq 1 = p_j$  for  $j \in B_1$ . Since  $\Lambda \notin V$ , we thus have  $Q \notin V$  as well.

For the second part of the theorem, let

$$m_1 = X_1^{a_1} \cdots X_n^{a_n}, \quad m_2 = X_1^{b_1} \cdots X_n^{b_n}$$

be monomials of the same degree such that  $m_1 > m_2$ . Then there is a smallest integer  $j$  such that  $a_i = b_i$  for  $i < j$  and  $a_j > b_j$ . Suppose  $m_2 \in J$ . We have to show that  $m_1 \in J$ .

Without loss of generality we may assume that  $b_j = a_j - 1$ . Then we have

$$a_{j+1} + \cdots + a_n + 1 = b_{j+1} + \cdots + b_n.$$

Since  $m_2 \in J$ , Lemma 4.2 implies that  $(L_{1,b_1+1}, \dots, L_{n,b_n+1}) \notin V$ . Hence the condition of Lemma 4.4 gives that  $(L_{1,a_1+1}, \dots, L_{n,a_n+1}) \notin V$  as well, so that again by Lemma 4.2 we have  $m_1 \in J$  as claimed.  $\square$

**Corollary 4.8.** *A configuration satisfying the condition of Lemma 4.3 is arithmetically Cohen-Macaulay.*

Now we can give the answer to the question posed at the beginning of this section, namely we identify precisely which configurations are the true liftings of Artinian monomial ideals, and which are the true liftings of Artinian lex-segment monomial ideals.

**Corollary 4.9.** *Let  $V$  be a configuration of linear varieties of codimension  $n$  in  $\mathbb{P}^{n+t-1}$ . Let  $A$  be a matrix of linear forms,*

$$A = \begin{bmatrix} L_{1,1} & L_{1,2} & L_{1,3} & \cdots & L_{1,N} \\ L_{2,1} & L_{2,2} & L_{2,3} & \cdots & L_{2,N} \\ \vdots & \vdots & \vdots & & \vdots \\ L_{n,1} & L_{n,2} & L_{n,3} & \cdots & L_{n,N} \end{bmatrix}$$

*such that  $L_{j,i} \in K[X_j, u_1, \dots, u_t]$  and such that the entries in any fixed row are pairwise linearly independent. Suppose that  $V$  is contained in the complete intersection  $X$  defined by the polynomials  $F_j = \prod_{i=1}^N L_{j,i}$ ,  $1 \leq j \leq n$ . Then  $V$  is the lifting (in the sense of Definition 2.3), via  $A$ , of an Artinian monomial ideal in  $n$  variables if and only if the condition of Lemma 4.3 holds. Furthermore,  $V$  is the lifting, via  $A$ , of an Artinian lex-segment monomial ideal if and only if in addition the condition of Lemma 4.4 holds.*

*Proof.* The proof of Theorem 3.4 (a) shows that  $(F_1, \dots, F_n)$  is a lifting of  $(X_1^N, \dots, X_n^N)$ , hence it is a complete intersection, and that  $X$  is reduced. Then Theorem 4.7 applies.  $\square$

**Remark 4.10.** The notion of a  $k$ -configuration of points, and related notions and applications, have received a good deal of attention in recent years (cf. e.g. [16], [14], [12], [10], [15], [11]). Our configurations above are somewhat related to these. In fact, the configurations of points arising from lex-segment ideals are special kinds of  $k$ -configurations, slightly more general than standard  $k$ -configurations but not as general as  $k$ -configurations. (It is easy to construct a lifting matrix to produce any standard  $k$ -configuration, but the need for so much collinearity prevents our obtaining an arbitrary  $k$ -configuration in this way.)

On the other hand, the configurations arising from arbitrary monomial ideals and satisfying the condition in Lemma 4.3 but not necessarily that in Lemma 4.4 are *not*  $k$ -configurations. The key is that in the definition of a  $k$ -configuration, for instance [10] Definitions 2.3 and 2.4, there is a strict inequality  $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ . The configurations produced by Lemma 4.3 only need to satisfy a weak inequality here (although note that in addition we still require the collinearity properties, so the weak inequality is not enough).

We will thus define a *weak  $k$ -configuration* of points in  $\mathbb{P}^n$  to be a finite set of points which satisfies the definition of a  $k$ -configuration as cited above, except that we require only  $\sigma(\mathcal{T}_i) \leq \alpha(\mathcal{T}_{i+1})$ . For points in  $\mathbb{P}^2$  this replacement of the strong inequality by the weak one was done in [14], and the result was called a *weak  $k$ -configuration*.

In what follows we would like to extend to higher dimension the notions of  $k$ -configurations and weak  $k$ -configurations of points in  $\mathbb{P}^n$ . The point of this is that our work on pseudo-lifting monomial ideals immediately shows that such configurations exist for any allowable Hilbert function.

**Definition 4.11.** A  *$k$ -configuration of dimension  $d$  linear varieties* ( $d \geq 0$ ) is an arithmetically Cohen-Macaulay generalized stick figure of dimension  $d$  whose intersection with a general linear space of complementary dimension is a  $k$ -configuration. A *weak  $k$ -configuration of dimension  $d$  linear varieties* is an arithmetically Cohen-Macaulay generalized stick figure of dimension  $d$  whose intersection with a general linear space of complementary dimension is a weak  $k$ -configuration.

Note that  $k$ -configurations (resp. weak  $k$ -configurations) of dimension  $d$  linear varieties can be produced as pseudo-liftings of Artinian lex-segment (resp. non lex-segment) monomial ideals. We can now describe the generalized stick figures of Corollary 3.7 somewhat more clearly.

**Corollary 4.12.** *Let  $\{c_i\}$  be a  $t$ -times differentiable  $O$ -sequence of dimension  $t$ . Then  $\{c_i\}$  is the Hilbert function of a strong  $k$ -configuration of dimension  $t - 1$  linear varieties. If  $\{c_i\}$  is not the “generic” Hilbert function then it is also the Hilbert function of a weak  $k$ -configuration which is not a  $k$ -configuration.*

*Proof.* The “generic” Hilbert function is the  $t$ -times differentiable Hilbert function of dimension  $t$  whose corresponding  $h$ -vector is

$$1 \quad n \quad \binom{n+1}{2} \quad \binom{n+2}{3} \quad \cdots \quad \binom{n+k}{k+1} \quad 0.$$

Any Artinian monomial ideal with this Hilbert function is forced to be lex-segment. In any other case one can find a monomial ideal which is not lex-segment.  $\square$

**Remark 4.13.** We conclude with an observation about the behavior of liaison under pseudo-lifting. Again let  $J$  be an Artinian monomial ideal, let  $N_j$  be the maximum power of  $X_j$  that occurs in a minimal generator of  $J$ , and let  $N$  be the maximum of the  $N_j$ . Suppose we lift  $J$  to an ideal  $I$  using the lifting matrix

$$A = \begin{bmatrix} L_{1,1} & L_{1,2} & L_{1,3} & \cdots & L_{1,N} \\ L_{2,1} & L_{2,2} & L_{2,3} & \cdots & L_{2,N} \\ \vdots & \vdots & \vdots & & \vdots \\ L_{n,1} & L_{n,2} & L_{n,3} & \cdots & L_{n,N} \end{bmatrix}.$$



Notice that some of the linear forms  $L_{j,i}$  may not be used (if  $i > N_j$ ). Form the matrix

$$A' = \begin{bmatrix} L_{1,N_1} & L_{1,N_1-1} & \cdots & L_{1,1} & \cdots \\ L_{2,N_2} & L_{2,N_2-1} & \cdots & L_{2,1} & \cdots \\ \vdots & \vdots & & \vdots & \\ L_{n,N_n} & L_{n,N_n-1} & \cdots & L_{n,1} & \cdots \end{bmatrix}.$$

Since  $J$  is an Artinian monomial ideal, it includes among its generators a power of each variable. Hence we have contained in  $J$  the complete intersection

$$\tilde{J} = (X_1^{N_1}, \dots, X_n^{N_n}).$$

Of course we can also lift  $\tilde{J}$  using the matrix  $A$ , and we obtain a complete intersection  $\tilde{I} \subset I$ . The amusing fact that emerges is that the residual ideal  $[\tilde{I} : I]$  is the pseudo-lifting of  $K$ , but using  $A'$  rather than  $A$ ! We leave the details to the reader. (See [20] for useful facts about liaison theory.)

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